Logarithmic bounds on Fujita's conjecture

X smooth projective variety, L line bundle.

Questions

1) When is L very ample?
2) When is L basepoint free?
3) When does L define a birational morphism
4) How to construct line bundles with properties (1) – (3).

Not easy: If C is a hyperelliptic curve of genus $g \gg 1$ then

$\varphi_{K_C} : C \longrightarrow \mathbb{P}^g$ 2:1 so K_C

is not very ample, however

$\deg(K_C) = 2g-2 \gg 1$

Matsusaka's big theorem Let L be ample.

There is $M_0 = M_0(L^n, K_X \cdot L^{n-1}, n)$

such that mL is very ample

for $m \geq M_0$.

Basepoint free theorem Let (X, Δ)

klt log pair D nef Cartier st

$\exists a$ $aD - (K_X + \Delta)$ big and nef

then $\exists b$ st cD is

basepoint free for all $c \geq b$.

In particular if $K_X + \Delta$ big and nef

then $K_X + \Delta$ is semi-ample.

Conjecture (Fujita 1985)

Let L be ample. Suppose

$K_X + tL$ is nef. Then

$m(K_X + tL)$ is basepoint free

for $m > n+1-t$.

Special case $t = (n+1)$.

$K_X + (n+1)L$ is nef by

the cone theorem.

Then $K_X + (n+1)L$ is basepoint free

Further conjectures

- $K_X + (n+2)L$ is very ample.
- $K_X + (n+2)L$ embeds X as projectively normal variety (Mukai)

- $K_X + (n+s)L$ separates s-jets.

Should we believe the conjecture?

Example $X = \mathbb{P}^n$ $L = \mathcal{O}(1)$

$K_X + (n+1)L = \mathcal{O}_X$

$K_X + (n+2)L = \mathcal{O}(1)$

Example $X \subseteq \mathbb{P}^{n+1}$ hypersurface of degree d. l be line.

$$l \cap X = P_1 \cup \cdots P_d$$

$p : Y \to X$ blowing up of $P_1 \cdots P_{d-1}$

$$L = p^* \mathcal{O}(1) \otimes \left(\sum -E_i\right)$$

ample and globally generated outside P_d.

$$K_Y \otimes L^n = p^* \mathcal{O}_X(d-2)\left(-\sum E_i\right)$$

basepoint at P_d.

Evidence for the conjecture

1) Curves : SD.

2) Surfaces : Reider 1988.

3) 3-folds : Ein - Lazarsfeld 1993

4) 4-folds : Kawamata 1997.
$m \geq 7$ (2017)

5) 5-folds : Ye-Zhu : $m \geq 6$ (2020)

6) Angehrn - Siu 1995 : quadratic bound.

7) Fujino, Mustata, Payne (1997-2004)
: toric case

Theorem (GL)

1) $K_X + mL$ is basepoint free for
$m \geq n(\log\log(n)+3)$

2) If $\dim X = 6$ then $m \geq 8$.

3) If $\dim X = 3$ $\quad K_X + 5L$ is birational.

<u>Recall Angehrn - Siu's method</u>

Fix $x \in X$.

$$0 \to \mathcal{I}_x \to \mathcal{O}_X \to \mathbb{C}_x \to 0$$

$$0 \to \mathcal{I}_x \otimes \mathcal{O}_X(K_X + mL) \to \mathcal{O}_X(K_X + mL) \to \mathbb{C}_x \to 0$$

$$H^0(X, K_X + mL) \to \mathbb{C} \to \underbrace{H^1(X, \mathcal{I}_x \otimes \mathcal{O}_X(K_X + mL))}$$

need to prove this is zero.

<u>Nadel</u>: $H^1(X, \mathcal{I}(X, \Delta) \otimes \mathcal{O}_X(K_X + mL)) = 0$

provided $mL - \Delta$ big and nef.

We want $\mathcal{I}(X, \Delta) = \mathcal{I}_x$.

L ample $\Rightarrow$ $L^n \geq 1$

$\exists D_1 \sim_{\mathbb{Q}} nL$ $\qquad$ $\mathrm{ord}_x D_1 \geq n$

$\Rightarrow$ $x \in \mathrm{Supp}\, \mathcal{J}(X, D_1) = Z_1$

$\dim Z_1 = d_1$

Let $y \in Z_1$ near x.

$\exists D_2' \sim_{\mathbb{Q}} d_1 L|_{Z_1}$ $\qquad$ $\mathrm{ord}_x D_2' \geq d_1$

$\Rightarrow$ $x \in \mathrm{Supp}\, \mathcal{J}(Z_1, D_2')$

$\Rightarrow$ $x \in \mathrm{Supp}\, \mathcal{J}(X, D_1 + D_2) = Z_2$

$X \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_s = \{x\}$

$x = \mathrm{Supp}\, \mathcal{J}(X, D_1 + D_2 + \cdots + D_s)$

We need $nL - D_1 - D_2 - \cdots - D_s$ big and nef

$m \geq \sum d_i$

It suffices $m \geq \dfrac{n(n-1)}{2}$

<u>Def</u> (Helmke 1997)

X smooth variety

(X, Δ) log pair, lc at x.

$f : Y \to X$ blowing up at x

with exceptional divisor E.

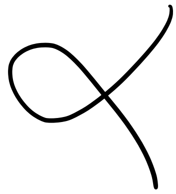
$$K_Y + \Gamma = f^*(K_X + \Delta)$$

The <u>local discrepancy of (X,Δ)</u>

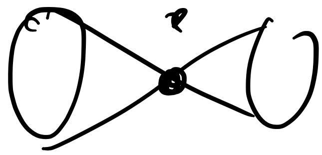
<u>over x</u> is

$b_x(X,\Delta) = \inf \{ b \mid (Y, \Gamma + bE)$ has a non klt center contained in $E \}$

Example

1) $X = \mathbb{P}^2 \qquad \Delta =$ 

$b_x(X, \Delta) = 0.$

2) $X = \mathbb{P}^3 \qquad \Delta =$ 

$$
\begin{array}{rcl}
f : Y = Bl_x \mathbb{P}^3 & \longrightarrow & X = \mathbb{P}^3 \\
E & \longrightarrow & x \\
\Delta' = \overline{\mathbb{F}_2} & \longrightarrow & \Delta = \overline{\mathbb{F}_2}
\end{array}
$$

$\Delta' \cap E = \mathbb{P}^1$

$K_Y + \Delta' = f^*(K_X + \Delta)$

$K_Y + \Delta' + bE \rightsquigarrow b = 1$

$b_x(\mathbb{P}^3, \Delta) = 1.$

Lemma (Helmke)

Let (X, Δ) be a log pair, lc at x, not klt

Let Z be the minimal lc center

at x.

Let Δ' be such that

- $Z \subsetneq \operatorname{Supp} \Delta'$
- $(X, \Delta + \Delta')$ is lc at x.

Then

$$b_x(X, \Delta + \Delta') \leq b_x(X, \Delta) - \operatorname{ord}_x(\Delta'|_Z)$$

Pf

Let $\pi : \tilde{X} \to X$ be the blowing up at x.

Let $g : Y \to \tilde{X}$ be a log resolution of $(\tilde{X}, \pi^*(\Delta + \Delta'))$

Let $f = \pi \circ g$

$$\begin{tikzcd} Y \arrow[d, "g"'] \arrow[dd, bend left=60, "f"] \\ \tilde{X} \arrow[d, "\pi"', "bl_x"] \\ X \end{tikzcd}$$

$\exists!$ exceptional component $E_0 \subseteq Y$ st

1) $a(X, \Delta, E_0) = -1$

2) $f(E_0) = Z$

By definition of $b_x(X, \Delta)$ there is a component $E_1 \subseteq Y$ st

$$a(Y, \underbrace{\Gamma}_{} + \underbrace{b_x(X,\Delta)\, E}_{}, E_1) = -1$$

By Shokurov connectedness

$$E_0 \cap E_1 \neq \phi.$$

$$\mathrm{ord}_{E_1} f^*\Delta' \underset{E_0 \cap E_1}{\overset{=}{\perp}} \mathrm{ord}_{E_0 \cap E_1} (f^*\Delta')|_{E_0}$$

$$= \mathrm{ord}_{E_0 \cap E_1} f^*(\Delta'|_{E_0})$$

$$= \mathrm{ord}_{E_0 \cap E_1} g^*(\pi^*(\Delta'|E_0))$$

$$= \mathrm{ord}_{E_0 \cap E_1} g^*\left(\mathrm{ord}_x(\Delta'|_Z) E + \Delta''\right)\Big|_{E_0}$$

$$\geq \mathrm{ord}_x \Delta'|_Z \cdot \mathrm{ord}_{E_0 \cap E_1} g^*E|_{E_0}$$

$$= \mathrm{ord}_x \Delta'|_Z \cdot \mathrm{ord}_{E_1} g^*E$$

$$\operatorname{ord}_{E_1} f^*\Delta' \geq \operatorname{ord}_x (\Delta'|_t) \cdot \operatorname{ord}_{\bar{E}_1} g^* \bar{E}$$

$$K_Y + \Theta = \pi^*(K_X + \Delta + \Delta')$$
$$= \pi^*(K_X + \Delta) + \pi^*\Delta'$$
$$= K_Y + \Gamma + \pi^*\Delta'$$

$$a(Y, \Gamma + \pi^*\Delta' + b_x(X, \Delta+\Delta')E, E_1) = -1$$

$$\operatorname{ord}_{E_1}(\pi^*\Delta' + b_x(X, \Delta+\Delta')E)$$

$$= \operatorname{ord}_{E_1} b_x(X, \Delta) E$$

$$\left(\mathrm{ord}_x \Delta'|_Z + b_x(X, \Delta+\Delta')\right) \cancel{\mathrm{ord}_{E_1} g^* Z}$$

$$\leq$$

$$b_x(X, \Delta) \cdot \cancel{\mathrm{ord}_{E_1} g^* Z}$$

$$b_x(X, \Delta + \Delta') \leq b_x(X, \Delta) - \mathrm{ord}_x \Delta'|_Z$$

Lemma X irreducible projective

D $\mathbb{Q}$-Cartier divisor.

$\exists D' \sim_{\mathbb{Q}} D$ st

$$\mathrm{ord}_x D' \geq \left(\frac{\mathrm{vol}(X, D)}{\mathrm{mult}_x X} \right)^{1/n}$$

Lemma

$$\mathrm{mult}_x Z \leq \binom{n - \lceil b_x \rceil}{n - d}$$

Example $X = \mathbb{P}^n$

$$\Delta = \sum_{i=1}^{n} H_i$$

$Z =$ union of d- dimensional strata

$$b_x = 0$$

$$\mathrm{mult}_x Z \leq \binom{n}{n-d}$$

Proof of Theorem

Step 1 $\quad L^n \geq 1$

$\exists D_1 \sim_{\mathbb{Q}} L$ st $\operatorname{ord}_x D_1 \geq 1$

let $t_1 = \operatorname{lct}(X, D_1, x)$

$t_1 \leq n$

$Z_1 = \operatorname{NKlt}(X, t_1 D, x)$

$d_1 = \dim Z_1$

$m_1 = \operatorname{mult}_x Z_1$

$b_1 = b_x(X, t_1 D) \leq n - t_1 \cdot \operatorname{ord}_x D_1$

$\leq n - t_1$.

$t_1 \leq n - b_1$

Step 2

$\exists D_2' \sim_{\mathbb{Q}} L|_{Z_1}$ st

$\mathrm{ord}_x D_2' \geq \left(\frac{1}{m_1}\right)^{d_1}$

let $t_2 = \mathrm{lct}(Z_1, D_2', x)$

$Z_2 = \mathrm{NKLT}(X, t_2 D_2', x)$

$d_2 = \dim Z_2$

$m_2 = \mathrm{mult}_x Z_2$

$b_2 = b_x(X, t_1 D_1 + t_2 D_2)$

$\leq b_1 - t_2 \, \mathrm{ord}_x D_2'$

$\leq b_1 - t_2 \cdot \left(\frac{1}{m_1}\right)^{d_1}$

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Goal: estimate $\sum t_i$

$$\sum t_i \le n - b_1 + (b_1 - b_2)\, m_1^{1/d_1}$$

$$+ \cdots$$

$$m_1 \le \binom{n - \lceil b_1 \rceil}{n - d_1}$$

$$\sum t_i \le n - b_1 + (b_1 - b_2) \binom{n - \lceil b_1 \rceil}{n - d_1}^{1/d_1} + (b_2 - b_3) \binom{n - \lceil b_2 \rceil}{n - d_2}^{1/d_2}$$

$$+ \cdots$$

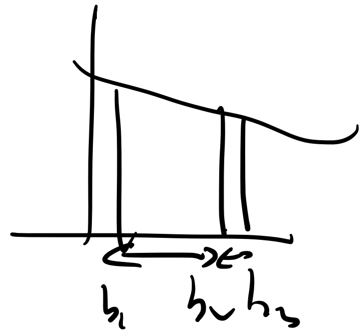
$$= \sum (b_i - b_{i+1}) \binom{n - \lceil b_i \rceil}{n - d_i}^{1/d_i}$$

$b_i \le d_i$

<u>Lemma</u> It suffices to choose

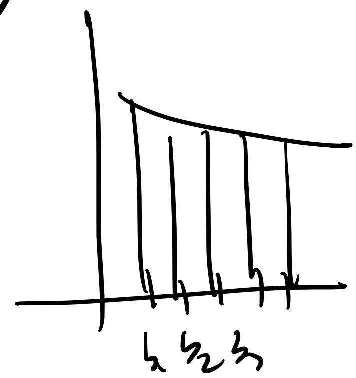
b_i integers.

$$\sum (b_i - b_{i+1}) \binom{n - \lceil b_i \rceil}{n - d_i}^{1/d_i}$$



$$\leq$$

$$\sum \binom{n-b}{n-d(b)}^{1/d(b)}$$



Goal: estimate

$$\binom{n-b}{n-d}^{1/d}$$

$$\binom{n-b}{n-d} = \frac{(n-b)\cdots(n-d+1)}{(b-d)!} \leq$$

$$\leq n^{d-b}$$

$$\geq \left(\frac{d-b}{e}\right)^{d-b}$$

$$\leq \left(\frac{en}{d-b}\right)^{d-b}$$

$\delta = d-b$

$$f(\delta) = \left(\frac{en}{\delta}\right)^{\frac{\delta}{b+\delta}}$$

$$\log \psi(\delta) = \frac{\delta}{b+\delta}\left(1 + \log n - \log \delta\right)$$

$$\frac{d}{d\delta} \log \psi(\delta) = \frac{b}{(b+\delta)^2}\left(1+\log n - \log \delta\right)$$

$$- \frac{1}{b+\delta}$$

$$\frac{d}{d\delta} = 0 \iff \cancel{b} + b\log n - b\log \delta$$

$$= \cancel{b} + \delta$$

$$\delta = b\left(\log n - \log \delta\right)$$

$$\delta = b\, W\!\left(\frac{n}{b}\right)$$

$$W\!\left(\frac{n}{b}\right) = \log n - \log W\!\left(\frac{n}{b}\right)$$

<u>Def</u> : Lambert's function

$W : \mathbb{R}^+ \to \mathbb{R}^+$ is the inverse of $f(x) = x e^x$

$$W(x)\, e^{W(x)} = x$$

$$\log W(x) + W(x) = \log x$$

$$\binom{n-b}{n-d}^{1/d} \leq e^{W\left(\frac{n}{b}\right)}$$

$$\sum \binom{n-b}{n-d(b)}^{1/d(b)} \leq \int e^{W\left(\frac{n}{x}\right)}\, dx$$

$$\sim n \log\log n.$$

$\square$